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Note

Orientable convexity, geodetic and hull numbers in graphs

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Abstract

We prove three results conjectured or stated by Chartrand and Zhang [European J. Combin. 21 (2000) 181–189] and Chartrand et al. [Discrete Appl. Math. 116 (2002) 115–126; Internat. J. Math. Math. Sci. 36 (2003) 2265–2275]: a connected graph has orientations with different geodetic numbers, orientations with different hull numbers, and, if there are no end-vertices, orientations with different convexity numbers. The proof of the first result is a correction of Chartrand and Zhang's proof, and allows for an easy proof of the second result. The third result says roughly that graphs without end-vertices can be oriented anti-transitively.

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The aim of this paper is to establish the following results, for every connected graph G with at least three vertices:

$$g^-(G) < g^+(G), \tag{1}$$

$$h^-(G) < h^+(G), \tag{2}$$

$$\text{con}^-(G) < \text{con}^+(G) \quad \text{iff } G \text{ has no end-vertices.} \tag{3}$$

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Results (2) and (3) were conjectured by Chartrand et al. [3,2], respectively. The first result was stated by Chartrand and Zhang [1, Theorem 2.5], but there was a gap in their proof. They independently noticed this gap, and an alternative proof was found, but the correction we present in Section 3 allows us to prove (1) and (2) simultaneously. We prove (3) in Section 2.

1. Preliminaries

Let $D = (V, A)$ be a digraph, and let u and v be vertices. A u – v *geodesic* is a dipath from u to v with the least possible number of arcs (so all u – v geodesics have the same length). The *closed interval* $I[u, v]$ consists of u , v , and every vertex that is on some u – v geodesic or on some v – u geodesic (note that there may be no directed path at all from u to v , or from v to u). For a set $S \subseteq V(D)$, we define $I[S] := \bigcup_{u,v \in S} I[u, v]$, and, for $k > 0$, $I^k[S] := I[I^{k-1}[S]]$, where $I^0[S] := S$.

A set S is *convex* if $S = I[S]$, that is, every geodesic between every two vertices of S lies in S . The *convex hull* $[S]$ of S is the smallest convex set containing S ; this is the intersection of all convex sets containing S , and also the limit of the sequence $S \subseteq I[S] \subseteq I^2[S] \subseteq \dots$.

A *hull-set* of D is a set $S \subseteq V$ for which $[S] = V$. If, moreover, $I[S] = V$, then S is a *geodetic set*. The *hull number* of D is

$$h(D) := \min\{|S| \mid S \text{ is a hull-set of } D\},$$

while the *geodetic number* of D is

$$g(D) := \min\{|S| \mid S \text{ is a geodetic set of } D\}.$$

For an undirected graph G , an orientation \vec{G} is a digraph obtained by giving each edge one of its two possible directions. The *lower* and *upper orientable hull numbers* are, respectively,

$$\begin{aligned} h^-(G) &:= \min\{h(\vec{G}) \mid \vec{G} \text{ is an orientation of } G\} \text{ and} \\ h^+(G) &:= \max\{h(\vec{G}) \mid \vec{G} \text{ is an orientation of } G\}. \end{aligned}$$

The *lower* and *upper orientable geodetic numbers* $g^-(G)$ and $g^+(G)$ are defined similarly.

Let v be a vertex in a digraph $D = (V, A)$. Its *in-* and *out-neighbourhood* are $N^-(v) := \{u \mid \vec{uv} \in A\}$ and $N^+(v) := \{w \mid \vec{vw} \in A\}$, respectively. Its *in-* and *out-degree* are $id(v) := |N^-(v)|$ and $od(v) := |N^+(v)|$, respectively. If, for every $u \in N^-(v)$ and every $w \in N^+(v)$, $\vec{uw} \in A$, then v is *extreme*. An (extreme) vertex v is a *source* if $N^-(v) = \emptyset$, and a *sink* if $N^+(v) = \emptyset$.

A graph that can be oriented so that every vertex is extreme is a *comparability* graph, or *transitively orientable* graph. A result that we will use repeatedly is the following, due to Chartrand et al. [2, Proposition 2.1; 3, Proposition 1.3]:

Proposition 1. *A vertex v is extreme iff, for every u and w in V , v is not an interior vertex of any u – w geodesic. Therefore, v is extreme iff $V - v$ is a convex set, iff v is contained in every hull-set and every geodetic-set.*

2. Orientable convexity numbers

If $D = (V, A)$ is a digraph, the *convexity number* $con(D)$ is the size of the largest convex set $C \subsetneq V$ (V itself is always convex). For an undirected graph G , $con^-(G)$ and $con^+(G)$ are the minimum and maximum convexity numbers over all orientations of G . We are interested in whether $con^-(G) < con^+(G)$.

By Proposition 1, D has an extreme vertex iff $con(D) = n - 1$, where n is the number of vertices. For any graph G , we can make an arbitrary vertex v extreme by orienting all incident edges away from v , so we always have $con^+(G) = n - 1$. Moreover, if G contains an end-vertex x , then in every orientation x is either a source or a sink; so in this case, $con^-(G) = n - 1$ too.

If G has no end-vertices, it is straightforward to find an orientation with no sources or sinks; the reader is encouraged to do so, and then try to generalise this to avoid all extreme vertices. We present a solution below.

Let some of the edges of G be oriented. A vertex incident to some oriented edge is an *or-vertex*, short for *oriented vertex*. Note that a vertex v is non-extreme iff there are arcs \overrightarrow{uv} and \overrightarrow{vw} , such that uw is either not present, or it is already oriented as \overrightarrow{uw} . No matter how the remaining undirected edges are oriented, v remains non-extreme.

Theorem 2. *A graph with minimum degree 2 can be oriented so that all its vertices are non-extreme. Thus, for a connected graph G with at least 3 vertices, $con^-(G) < con^+(G)$ iff G has no end-vertices.*

Proof. Since G has minimum degree 2, it contains a cycle. Find a maximal set M of edge-disjoint *chordless* cycles, and orient their edges to make them directed cycles. We claim that every or-vertex v is now non-extreme. If v is on a triangle uvw in M , then \overrightarrow{uv} , \overrightarrow{vw} and \overrightarrow{wu} are all arcs. Otherwise, v is on a chordless cycle in M of length at least 4, with neighbours, say, u and w , where $uw \notin E(G)$ but \overrightarrow{uv} and \overrightarrow{vw} are arcs.

We now show that, if there are unoriented vertices, we can orient one or more while maintaining the property that all or-vertices are non-extreme.

Any unoriented vertex u must be on a path u_0, \dots, u_{r+1} joining distinct or-vertices u_0 and u_{r+1} (because the graph has minimum degree at least 2, and our initial set M of edge-disjoint cycles was chosen to be maximal). Taking r to be as small as possible ensures that the internal vertices u_1, \dots, u_r are all unoriented. Directing the path as $\overrightarrow{u_0u_1}, \dots, \overrightarrow{u_ru_{r+1}}$ ensures that u_1, \dots, u_r all have positive in- and out-degree. Moreover, if $r > 1$, then, for $1 \leq i \leq r$, $u_{i-1}u_{i+1} \notin E(G)$, and thus u_i is non-extreme.

If $r = 1$, then we might have to orient differently if u_0u_2 is an edge of G . Since u_0u_1 and u_1u_2 are unoriented, and M was maximal, u_0u_2 must be oriented, say as $\overrightarrow{u_0u_2}$. Now orienting u_0u_1 and u_1u_2 as $\overleftarrow{u_0u_1}$ and $\overleftarrow{u_1u_2}$, ensures that u_1 is on a directed triangle and is thus non-extreme. \square

3. Orientable geodetic and hull numbers

Chartrand and Zhang's proof of (1) essentially found a vertex v_1 , and orientations D_1 and D_2 of G , such that if S is a hull-set in D_2 , then $I_{D_2}[S] \subseteq I_{D_1}[S - v_1]$ (this is Claim 1 in our own proof). Moreover, v_1 was a source in D_2 , and was thus contained in every hull-set. By taking S to be a minimum geodetic set for D_2 , we immediately get $g^-(G) < g^+(G)$. With slightly more work (Claim 2 in our proof), we also get $h^-(G) < h^+(G)$, proving Conjecture 3.10 of [3].

Chartrand and Zhang stated their result only for orientable geodetic numbers, as they did not include Claim 2. Moreover, they oriented $G[U]$ arbitrarily (where U is a set of vertices specified in the proof, and $G[U]$ is the subgraph of G induced by U). The path of length four (for example) shows that this does not always work, and their alternative proof did not extend to showing $h^-(G) < h^+(G)$. There is, however, an orientation of $G[U]$ that will rescue the original proof, as we show below.

Theorem 3. *For any connected graph G with at least three vertices, $g^-(G) < g^+(G)$ and $h^-(G) < h^+(G)$.*

Proof. If G is a complete graph with vertices v_1, \dots, v_n , we first orient G transitively (that is, $v_i \rightarrow v_j$ iff $i < j$). Since every vertex is extreme, this orientation shows that $g^+(G) = n = h^+(G)$. Reversing the orientation of $v_1 v_2, \dots, v_{n-1} v_n$ makes $\{v_1, v_2\}$ a geodetic set; thus $g^-(G) = 2 = h^-(G)$.

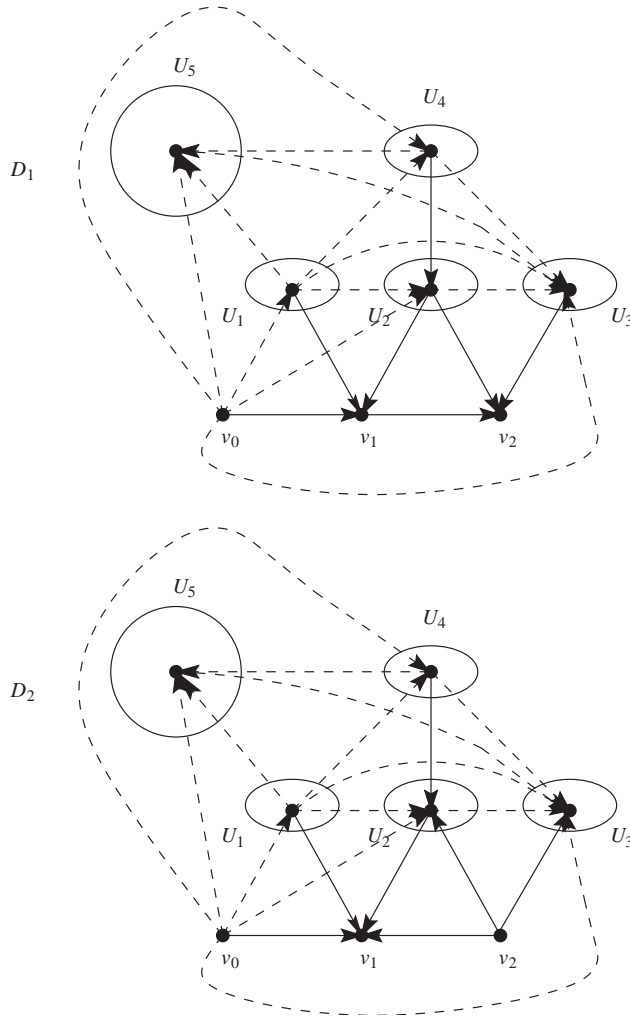
If G is not complete, then we can find vertices v_0, v_1, v_2 that induce a path of length two. Fig. 1 shows all the adjacencies (solid lines) and possible adjacencies (dashed lines) in G , where the U_i 's are defined as follows. For a set $C \subseteq V(G)$, $N(C)$ is the set $\{v \in V \mid \exists c \in C, vc \in E\}$.

$$\begin{aligned} U &:= V(G) \setminus \{v_0, v_1, v_2\}, \\ U_1 &:= U \cap (N(v_1) \setminus N(v_2)), \\ U_2 &:= U \cap (N(v_1) \cap N(v_2)), \\ U_3 &:= U \cap (N(v_2) \setminus N(v_1)), \\ U_4 &:= (U \cap N(U_2)) \setminus (U_1 \cup U_2 \cup U_3), \text{ and} \\ U_5 &:= U \setminus (U_1 \cup U_2 \cup U_3 \cup U_4). \end{aligned}$$

Let D_2 be the digraph obtained by orienting G as follows.¹ We orient an edge xy from x to y if one of the following conditions holds:

$$\begin{array}{ll} x \in \{v_0, v_2\} & \text{or} \quad y = v_1, \\ x \in U_1 & \text{and} \quad y \in U \setminus U_1, \\ x \in U_4 & \text{and} \quad y \in U_2, \\ x \in U \setminus U_3 & \text{and} \quad y \in U_3. \end{array}$$

¹ The labeling is chosen to be consistent with Chartrand and Zhang, but I prefer to describe D_2 before D_1 .

Fig. 1. The orientations D_1 and D_2 of G .

All other edges join vertices within the same U_i , and are oriented arbitrarily. It can be checked that the conditions are self-consistent. We obtain D_1 from D_2 by reversing the orientation of the arcs incident to v_2 .

Claim 1. *If S is a hull-set in D_2 , then $I_{D_2}[S] \subseteq I_{D_1}[S - v_1]$.*

Since S is a hull-set for D_2 , it must contain the extreme vertices v_0 and v_2 . In D_1 , v_1 is on a v_0 – v_2 geodesic, and is thus in $I_{D_1}[S - v_1]$. So $S \subseteq I_{D_1}[S - v_1]$.

Consider, therefore, a vertex $w \in I_{D_2}[S] \setminus S$; note that $w \in U$. This vertex must be an internal vertex of an a – b geodesic P in D_2 , for some a and b in S . If a and b are both in U ,

then $V(P) \subseteq U$; since the orientation of $G[U]$ is the same in D_1 as in D_2 , P is present in D_1 . Moreover, the a – b dipaths in D_1 are just the a – b dipaths in D_2 , so P is still a *shortest* a – b dipath. Since a and b are in $S - v_1$, $w \in I_{D_1}[S - v_1]$.

If $a = v_0$, then $b \neq v_1$ (since the only v_0 – v_1 geodesic is $\overrightarrow{v_0 v_1}$), and clearly $b \neq v_2$, so $b \in U$. Moreover, the a – b dipaths do not use v_1 or v_2 , so D_1 contains all the a – b dipaths of D_2 , and no others; thus P is still an a – b geodesic in D_1 . As above, a and b are in $S - v_1$, so $w \in I_{D_1}[S - v_1]$.

If $a = v_2$, then b must be in $N(v_2)$; but then the unique a – b geodesic in D_2 is \overrightarrow{ab} , with no internal vertices.

If $b = v_1$, then I claim that P must have vertices awv_1 , with $a \in U_4$ and $w \in U_2$. To see this, note that a cannot be in $N(v_1)$, as otherwise the only a – v_1 geodesic is $\overrightarrow{av_1}$. Moreover, there are no dipaths from $U_3 \cup U_5$ to v_1 , so a must be in U_4 . By definition of U_4 , and by the choice of orientation, there is a (directed) path of length two from a to v_1 , so every a – v_1 geodesic has length two. The internal vertex must be adjacent to v_1 , but cannot be in U_1 (by choice of orientation), so it must be in U_2 .

Since a is in U_4 , it is not adjacent to v_2 ; but in D_1 there is a directed path awv_2 , and this is therefore an a – v_2 geodesic. Since a and v_2 are in $S - v_1$, w is in $I_{D_1}[S - v_1]$.

Claim 2. *If S is a hull-set in D_2 , then $I_{D_2}^\ell[S] \subseteq I_{D_1}^\ell[S - v_1]$ for any $\ell \geq 1$.*

We proceed by induction on ℓ , the base case $\ell = 1$ following from Claim 1. Now for $\ell > 1$,

$$\begin{aligned} I_{D_2}^\ell[S] &= I_{D_2}[I_{D_2}^{\ell-1}[S]] \subseteq I_{D_1}[I_{D_2}^{\ell-1}[S] - v_1] \\ &\subseteq I_{D_1}[I_{D_1}^{\ell-1}[S - v_1] - v_1] \subseteq I_{D_1}[I_{D_1}^{\ell-1}[S - v_1]] = I_{D_1}^\ell[S - v_1]. \end{aligned}$$

The first containment follows from Claim 1 applied to the hull-set $I_{D_2}^{\ell-1}[S]$, while the second follows from the inductive hypothesis.

If S is a hull-set for D_2 , then $I_{D_2}^k[S] = V$, for some k . By Claim 2, $I_{D_1}^k[S - v_1] = V$, so $S - v_1$ is a hull-set for D_1 . In particular, v_1 is a sink in D_2 , so it is contained in S , and taking S to be a minimum hull-set for D_2 we have

$$h^-(G) \leq h(D_1) \leq |S - v_1| < |S| = h(D_2) \leq h^+(G).$$

If S is a (minimum) geodetic set for D_2 , then we can take $k = 1$, so $S - v_1$ is a geodetic set for D_1 and we have $g^-(G) < g^+(G)$. \square

Since every geodetic set is a hull-set, we have $h(D) \leq g(D)$ for every digraph D . For an undirected graph G we therefore have $h^-(G) \leq g^-(G)$ and $h^+(G) \leq g^+(G)$, and together with Theorem 3 this leaves five possibilities:

$$h^- = g^- < h^+ = g^+, \quad (4)$$

$$h^- = g^- < h^+ < g^+, \quad (5)$$

$$h^- < g^- < h^+ = g^+, \quad (6)$$

$$h^- < g^- = h^+ < g^+, \quad (7)$$

$$h^- < h^+ < g^- < g^+. \quad (8)$$

Chartrand et al. identified many infinite classes of graphs for which (4) holds, including trees, cycles and complete bipartite graphs. For complete bipartite graphs $K_{s,t}$ with $s \geq t \geq 2$ [1, Proposition 3.8], and for transitively orientable graphs with a Hamiltonian path, we have $h^-(G) = g^-(G) = 2 < n = h^+(G) = g^+(G)$. If T is a tree with k end-vertices, then $h^-(T) = g^-(T) = k < |V(T)| = h^+(T) = g^+(T)$, while $h^-(C_{2n+1}) = g^-(C_{2n+1}) = 2 < 2n = h^+(C_{2n+1}) = g^+(C_{2n+1})$. We leave the realisability of (5)–(8) as open problems.

Problem 4. Find infinite classes of graphs for which (5), (6) or (7) hold. Are there (infinitely many) graphs for which (8) holds?

Note that (8) cannot hold for graphs G for which there is an orientation \vec{G} such that $g(\vec{G}) = h(\vec{G})$. However, there are probably many graphs for which no such orientation exists.

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